ALGORITHM FOR HYBRID MODELING OF NONSTEADY-STATE THERMAL CONDUCTIVITY PROBLEMS

V. V. Mikhailov

UDC 536.24:681.3

An algorithm is presented for solution of linear and nonlinear nonsteady-state thermal conductivity problems with hybrid computation equipment.

At the present time ever wider use is being made of hybrid (analog—digital) computer systems for processing of thermophysical data. In connection with this, there arises the problem of hybrid modeling of forward and converse thermal conductivity problems and the development of corresponding computation algorithms for use as problem solving tools.

When analog and digital computation devices are used simultaneously, special attention must be given to the problem of rational division of the problem into "analog" and "digital" portions, as well as choice of the rate of data exchange between the components. These problems are closely related to the value of uncertainty in the modeling, which consists of the uncertainties of the individual devices composing the hybrid system, and uncertainties generated by simultaneous use of the devices with differing forms of data representation. Possible sources of such uncertainties were considered in [1, 2]. We will only note that uncertainties of the second type can be reduced markedly by rational division of the problem into analog and digital portions. On the other hand, it is necessary to consider the fact that highest efficiency is achieved with use of analog equipment when continuous operations are performed on functions of a single variable, and also when the Cauchy problem is solved for ordinary differential equations.

One of the desirable applications of hybrid modeling is solution of converse problems in the extremal formulation [3], for which a large number of "digital" algorithms have been developed on the basis of iteration techniques for minimization of the mean square discrepancy. As a rule the most cumbersome part of such algorithms is the procedure for solving the thermal conductivity boundary problems and calculating the gradient of the discrepancy by solving the boundary problem for the conjugate variable. If integration of these problems is assigned to the analog portion of the hybrid computation system, a significant increase in speed of the iteration algorithms is possible.

We will consider one approach to construction of hybrid modeling algorithms for such boundary problems using the following nonsteady state thermal conductivity problem as an example:

$$C(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T}{\partial x}\right],\ 0<x<b,\ t_0<t\leqslant t_p, \tag{1}$$

$$T(x,\ t_0)=\varphi(x),\ \ 0\leqslant x\leqslant b, \tag{2}$$

$$\alpha_1 T(0,\ t)-\beta_1\lambda(T)\frac{\partial T(0,\ t)}{\partial x}=u_1(t), \tag{3}$$

$$\alpha_2 T(b,\ t)-\beta_2\lambda(T)\frac{\partial T(b,\ t)}{\partial x}=u_2(t), \tag{4}$$

where α_i, β_i are nonnegative numbers such that $\alpha_i + \beta_i > 0$, $i = 1, 2$.

Representing the time variable t by a set of discrete points t_k equally spaced by an amount τ, $k=\overline{0, l}$, and approximating the time derivative by the expression $\partial T/\partial t \simeq (T_k - T_{k-1})/\tau$, the problem of Eqs. (1)-(4) at time t_k, $k=\overline{1, l}$, can be written

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 49, No. 6, pp. 950-954, December, 1985. Original article submitted May 17, 1985.

$$C^{(s)}(x)\frac{T_k^{(s)}-T_{k-1}}{\tau}=\frac{\partial}{\partial x}\left[\lambda^{(s)}(x)\frac{\partial T^{(s)}}{\partial x}\right],\ 0<x<b, \tag{5}$$

$$T_{k-1}(x)=\varphi_k(x),\ 0\leqslant x\leqslant b, \tag{6}$$

$$\alpha_1 T_k^{(s)}(0)-\beta_1\lambda^{(s)}(0)\frac{\partial T_k^{(s)}(0)}{\partial x}=u_1^k, \tag{7}$$

$$\alpha_2 T_k^{(s)}(b)-\beta_2\lambda^{(s)}(b)\frac{\partial T_k^{(s)}(b)}{\partial x}=u_2^k, \tag{8}$$

where s is the iteration number for the unknown profile $T_k(x)$; $\lambda(x)=\lambda[T(x)]$; $C(x)=C[T(x)]$.

The spatial variable x remains continuous and is modeled by "machine" time in the analog portion of the hybrid computer system.

It is obvious that such a discretization scheme is applicable for both linear and nonlinear problems with a large number of nonlinear functions. Moreover, with regard to the apparatus required, a significantly smaller number of analog elements is required than in the method based on discretization of the spatial variable.

Omitting the iteration number (s) and the time step number k, the problem of Eqs. (5)-(8) can be rewritten in the form of a two-point boundary problem for a second order differential equation

$$\frac{d}{dx}\left[\lambda(x)\frac{dT}{dx}\right]-q(x)T=f(x), \tag{9}$$

$$\alpha_1 T(0)-\beta_1\lambda(0)\frac{dT(0)}{dx}=u_1, \tag{10}$$

$$\alpha_2 T(b)-\beta_2\lambda(b)\frac{dT(b)}{dx}=u_2, \tag{11}$$

where $q(x)=C(x)/\tau$; $f(x)=-C(x)\varphi(x)/\tau$.

Thus, the original problem of Eqs. (1)-(4) has been reduced to successive solution at each k-th point in time of boundary problem (9)-(11).

To construct a stable computation process we write two-point boundary problem (9)-(11) in the form of a set of problems in Cauchy formulation. To do this we employ the factorization method of [2, 4], which consists of expanding the second order operator in first order operator, each of which is stable for a corresponding integration direction.

Following this method, the operator of Eq. (9)

$$L=\frac{d}{dx}\left[\lambda\frac{d}{dx}\right]-q$$

will be represented as a product of two operators:

$$L=L_1L_2(T)=\left[\Psi\frac{d}{dx}-\xi\right]\left[\zeta\lambda\frac{d}{dx}-\eta\right]T=f, \tag{12}$$

where the functions Ψ, ξ, ζ and η can be determined in the solution process.

By comparing Eq. (12) with the original equation, after simple transformations and use of the expression $\zeta+\eta=1$, in place of Eqs. (9)-(11) we can write the Cauchy problem for three first order equations:

$$\frac{d\eta}{dx} = -\frac{\eta^2}{\lambda} + q(1-\eta)^2, \ \eta(0) = \frac{\alpha_1}{\alpha_1+\beta_1}, \tag{13}$$

$$\frac{dy}{dx} = -\frac{\eta y}{\lambda} + (1-\eta)(f-qy), \ y(0) = -\frac{u_1}{\alpha_1+\beta_1}, \tag{14}$$

$$\frac{dT}{dx} = \frac{1}{\lambda(1-\eta)}[\eta T + y], \ T(b) = \frac{u_2[1-\eta(b)] + \beta_2 y(b)}{\alpha_2 - \eta(b)[\alpha_2+\beta_2]}. \tag{15}$$

In contrast to the equations presented in [4], the structure of differential equations (13)-(15) is independent of the form of boundary condition (10) (of the quantities α_1 and β_1).

Integration of Eq. (15) in the reverse direction leads to individual analog modeling of Eqs. (13), (14), and (15). In connection with this, provisions must be made for recall of the solutions $\eta(x)$ and $y(x)$.

In order to avoid these difficulties, we will represent the solution of Eq. (15) as a sum of the solution of the corresponding homogeneous equation and a particular solution of the nonhomogeneous equation, i.e.,

$$T(x) = T(b)\exp\left[-\int_x^b \frac{\eta}{\lambda(1-\eta)}dt\right] - \int_x^b \left[\frac{y}{\lambda(1-\eta)}\exp\left[-\int_x^t \frac{\eta}{\lambda(1-\eta)}d\tau\right]\right]dt$$

or

$$T(x) = \left\{T(0) + \int_0^x \left[\frac{y}{\lambda(1-\eta)}\exp\left[-\int_0^t \frac{\eta}{\lambda(1-\eta)}d\tau\right]\right]dt\right\} \Big/ \exp\left[-\int_0^x \frac{\eta}{\lambda(1-\eta)}dt\right]. \tag{16}$$

Denoting by $G(x)$ the solution of the homogeneous equation

$$\frac{dG}{dx} = -\frac{\eta}{\lambda(1-\eta)}G, \ G(0) = 1, \tag{17}$$

and by $Z(x)$ the solution of the equation

$$\frac{dZ}{dx} = -\frac{Z}{\lambda(1-\eta)}G, \ Z(0) = 0, \tag{18}$$

in the interval $[0, x]$, we may write the solution of Eq. (16) in the form

$$T(x) = [T(0) - Z(x)]/G(x),$$

or, with consideration of the fact that $T(0) = T(b)G(b) + Z(b)$:

$$T(x) = [T(b)G(b) + Z(b) - Z(x)]/G(x). \tag{19}$$

As a result, the solution of the two-point boundary problem (9)-(11) can be obtained by analog modeling of the Cauchy problem for four first order differential equations:

$$\frac{d\eta}{dx} = -\frac{\eta^2}{\lambda} + q(1-\eta)^2, \ \eta(0) = \frac{\alpha_1}{\alpha_1+\beta_1}, \tag{20}$$

$$\frac{dy}{dx} = -\frac{\eta y}{\lambda} + (1-\eta)(f-qy), \ y(0) = -\frac{u_1}{\alpha_1+\beta_1}, \tag{21}$$

$$\frac{dG}{dx} = -\frac{\eta}{\lambda(1-\eta)}G, \ G(0) = 1, \tag{22}$$

$$\frac{dZ}{dx} = -\frac{Z}{\lambda(1-\eta)}G, \ Z(0) = 0. \tag{23}$$

Upon completion of integration the second relationship of Eq. (15) is used to calculate the quantity $T(b)$, and Eq. (19) is used to calculate $T(x)$. Then a transition is accomplished to the solution of Eqs. (9)-(11) at the subsequent iteration or subsequent moment in time.

To estimate its accuracy the algorithm considered was realized in digital form. Results of the calculations were compared to analytical solutions for the case of an infinite plate, as presented in [5]. The closeness of the numerical solution to the exact one was determined by the value of the step in the finite difference approximation of the time derivative and the step in the discrete representation of the solution $T(x)$.

NOTATION

t, time; t_0, t_p, beginning and end of time interval; τ, discretization step in time; x, spatial variable; b, thickness; $T(x, T)$, temperature; $C(T)$, volume heat capacity; $\lambda(T)$, thermal conductivity coefficient; $u_1(t)$, $u_2(t)$, boundary functions.

LITERATURE CITED

1. I. M. Vitenberg, Programming Analog Computers [in Russian], Mashinostroenie, Moscow (1972).
2. I. M. Vitenberg and R. L. Tankelevich, Algorithmic Use of Analog Computers [in Russian], Énergiya, Moscow (1976).
3. O. M. Alifanov, "Some questions on solution of converse thermal conductivity problems and automated data processing in thermophysical research," Inzh.-Fiz. Zh., 39, No. 2, 211-219 (1980).
4. I. Babushka, É. Vitasek, and M. Prager, Numerical Methods for Solution of Differential Equations [Russian translation], Mir, Moscow (1969).
5. A. V. Lykov, Theory of Thermal Conductivity [in Russian], Vysshaya Shkola, Moscow (1967).